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THE PHOTON SCATTERING MATRIX ELEMENT FOR TRANSVERSE ELECTRIC OU--ETC(U)

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THE PHOTON SCATTERING MATRIX ELEMENT FOR TRANSVERSE ELECTRIC OUTGOING PHOTONS

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This report is a supplement to Second Order Radiative Effects Using Dirac Propagators and Vector Harmonic Phontons (Ph.D. dissertation by Harald O. Dogliani, University of Colorado, 1976).

The matrix element for photons scattering from bound K-shell electrons is presented. It is assumed that only magnetic multipoles are important and, hence, only transverse electric mode vector harmonic, outgoing photons are assumed. Electric multipole moments are accounted for in the Ph.D. dissertation.

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CONTENTS

INTRODUCTION

This report is a supplement to Second Order Radiative Effects Using Dirac Propagators and Vector Harmonic Photons, henceforth referenced as SORE, by Harald O. Dogliani, Ph.D. dissertation, University of Colorado, 1976. In SORE the scattering of circularly polarized photons by K-shell atomic electrons is treated using Dirac propagators representing intermediate state electrons and using a vector harmonic expansion for the outgoing photon fields. Since electric dipole transitions are usually assumed to dominate such a scattering process, only electric multipoles were treated in SORE by considering transverse magnetic, TM, outgoing photons and by neglecting transverse electric, TE, modes.

The analytic representation of the differential scattering cross section for photons scattering off K-shell electrons with outgoing vector harmonic photons is given by

$$d\sigma = \frac{2RV}{ch} \sum_{\ell,m} |M|^2 \, dk. \qquad 1$$

The differential scattering cross section, $d\sigma$, is a differential with respect to the final photon wave number k. The speed of light is c and V is the interaction volume with R being the radius of a spherical volume containing the outgoing photon of wave number k with angular momentum ℓ and projection m. Planck's constant is h. The matrix element for the interaction is M.

As a supplement to SORE, the matrix element for TE mode photons is presented in this report. By combining the TE mode matrix element presented in this report with the TM mode contributions presented in SORE, one can more properly treat photon scattering events in those situations where higher order multipole transitions are significant. The relevant terms for including TE mode scattering are presented in the appendix of this report. In order to calculate the matrix element for K-shell photon scattering, Equations 2, 3 and 5 must be combined with Equations 18, 19, 21 and 22 in the appendix.

A complete discussion of the vector harmonic expansion in terms of TE and TM modes is contained in SORE. Also included in SORE is a discussion on how a typical TE or TM contribution to the matrix element is derived. Finally, it should be emphasized that scattering matrix elements were developed using Dirac propagators for the intermediate electron states (see Sore). The use of single-electron, Dirac propagators with the use of vector harmonic outgoing photons results in expansions in terms of discrete exchanges of angular momenta. As is indicated in SORE, this may be of significant utility in certain scattering problems where total cross sections are needed.

TRANSVERSE ELECTRIC MATRIX ELEMENTS

The TE matrix element, averaged over initial electron spins, is given by

$$|M|^2 = \tfrac{1}{2}\{|M_1|^2 + |M_2|^2\} \qquad 2$$

where

$$M_1 = \sum_{m',\ell',p'} C(kk_o\ell m)\,(I_a\,(1,k\ell m{:}TE) + I_e(1,k\ell m{:}TE)),$$

$$M_2 = \sum_{m',\ell',\rho'} C(kk_o\ell m)(I_a(2,k\ell m{:}TE) + I_e\,(2,k\ell m{:}TE)),$$

$$C(kk_o\ell m) = he^3c\,(-1)^m\,[(2\omega_k)/(\ell(\ell+1)RV\omega_{k_o})]^{\frac{1}{2}}\,. \qquad 3$$

The following symbols are used:

k_o	incoming photon wave number
k	outgoing photon wave number
ω_{k_o}	incoming photon angular frequency
ω_k	outgoing photon angular frequency
e	charge of an electron
ℓ	outgoing photon angular momentum quantum number
m	outgoing photon angular momentum projection quantum number
m',ℓ',p'	intermediate state (propagator) quantum numbers (see SORE)
λ	incoming photon circular polarization $\lambda = 1,2,$
i	$\sqrt{-1}$. 4

The $I_a(\lambda, k\ell m:TE)$ and $I_e(\lambda, k\ell m:TE)$ must be evaluated for the absorption first and emission first contributions to M_1 and M_2 for incoming photon polarization λ. They are evaluated as follows:

$$I_a(1,k\ell m:TE) = \sum_{i=1}^{2} YATE(i)\ \delta_{m',3/2}\ \delta_{m,-m''+3/2}\ WATEU(i)$$

$$+ \sum_{i=3}^{6} YATE(i)\ \delta_{m',1/2}\ \delta_{m,-m''+1/2}\ WATED\ (i),$$

$$I_a(2,k\ell m:TE) = \sum_{i=1}^{2} YATE(i)\ \delta_{m',-3/2}\ \delta_{m,-m''-3/2}\ WATED(i)$$

$$+ \sum_{i=3}^{6} YATE(i)\ \delta_{m',1/2}\ \delta_{m,-m''+1/2}\ WATED(i),$$

$$I_e(1,k\ell m:TE) = \sum_{n=o}^{\infty} i^n(2(2n+1))^{\frac{1}{2}} \sum_{j=1}^{4} YETE(i)$$

$$[\delta_{m',-m+1/2}\ \delta_{m,-m''+3/2}\ WETEU(i)$$

$$+ \delta_{m',-m-1/2}\ \delta_{m,-m''+1/2}\ WETED(i)],$$

$$I_e(2,k\ell m:TE) = \sum_{n=0}^{\infty} i^n\ (2(2n+1))^{1/2} \sum_{j=1}^{4} YETE(i)$$

$$[\delta_{m',-m+1/2}\ \delta_{m,-m''-1/2}\ WETEU(i+4)$$

$$+ \delta_{m',-m-1/2}\ \delta_{m,-m''-3/2}\ WETED(i+4)]. \qquad 5$$

The final electronic states are specified by the double primed quantum numbers ℓ'',m'' and p''. The integrals YETE and YATE, and the coefficients WETED, WETEU, WATED, and WATEU are provided in the appendix.

APPENDIX

Enumeration of the YETE'S, YATE'S, WETE'S and WATE'S

The functions required to evaluate the matrix elements describing photons scattering from K-shell electrons are presented in this section. Only the magnetic multipoles (TE modes) are accounted for in this scheme. These functions are inserted into Equation 5.

The following mnemonics are used in the definitions of the constants:

1) all constants beginning with the letter "Y" represent radial integrals;

2) all constants whose second letter is "A" are encountered in the "absorption first" matrix element;

3) similarly, all constants whose second letter is "E" are encountered in the "emission first" case;

4) all constants end with "TE" since this set considers TE modes only.

The $j_n(k_o r)$ are spherical Bessel functions of order n, whereas the lower case g(r) and f(r) are the radial wavefunctions for the K-shell electron initial states - See SORE Equations (3-54) and (3-73). The final electronic state radial wave functions are represented by G(r) and F(r) as introduced in Equation (3-58), SORE. The following functions are also used:

$$I(j,k:\ell,m:n,i) = \int Y^*_{jk}(\theta,\phi)\, Y_{\ell m}(\theta,\phi)\, Y_{ni}(\theta,\phi)\, d\Omega, \qquad 6$$

Where the $Y_{jk}(\Theta,\phi)$ are the usual spherical harmonics.

The C(jk:p) are given by

$$C\,(jk:p) = C^{jp}\, C^{kp} / \, chy^{p} \qquad 7$$

where the C^{jp} are

$$C^{1p} = [(\ell + 1/2 + pm)/(2\ell + 1)]^{1/2},$$

$$C^{2p} = -p[(\ell + 1/2 - pm)/(2\ell + 1)]^{1/2},$$

$$C^{3p} = -[(\ell + 1/2 + p - pm)/(2\ell + 2p + 1)]^{1/2},$$

$$C^{4p} = -p[(\ell + 1/2 + p + pm)/(2\ell + 2p + 1)]^{1/2}, \quad 8$$

and y^p is related to the Wronskian of the radial propagators, S(r) and R(r), where

$$y^p = S^{i*}(r)\, R^r(r) - S^r(r)\, R^{i*}(r). \quad 9$$

The $S^r(r)$, $S^i(r)$, $R^r(r)$ and $R^i(r)$ are the regular and irregular S and R solutions, respectively, to the following coupled differential equation

$$(mc^2 + V - E)\, S(r) - c\hbar\left(\frac{d}{dr} + \frac{K}{r}\right) R(r) = 0,$$

$$(-mc^2 + V - E)\, R(r) + c\hbar\left(\frac{d}{dr} - \frac{k}{r}\right) S(r) = 0. \quad 10$$

In Equation 10, k is a non zero, integer quantum number, E the propagator energy and V the potential. The radial propagators, RR(p'ri:rr'), RS(...) and SS(...), are given by

$$RR(p'ri:rr') = R^r(r)R^{i*}(r')/rr',$$

$$RS(p'ri:rr') = R^r(r)S^{i*}(r')/rr',$$

$$SS(p'ri:rr) = S^r(r)S^{i*}(r')/rr'. \quad 11$$

If the electron is initially in a spin up state represented by $\psi_\beta(\bar{r}')$ then *

$$\psi_{\beta_1}(\bar{r}') = \begin{bmatrix} C^{1+}\, Y_{00}(\theta',\phi')\, g(r') \\ 0 \\ C^{3+}\, Y_{10}(\theta',\phi')\, if\,(r') \\ C^{4+}\, Y_{11}(\theta',\phi')\, if\,(r) \end{bmatrix} \quad 12$$

*Bethe, M.A. and Salpeter, E.E., Quantum Mechanics of One- and Two-Electron Atoms (Springer-Verlag, Berlin, 1957) p. 64.

$$WATEU(5) = AC[C^{1P''} \{ C(42:p')\ ((\ell+m''+1/2)\ (\ell-m''+1/2))^{1/2}$$

For an arbitrary, spherically symmetric potential the radial wave functions, $g(r)$ and $f(r)$, must usually be obtained by numerical means. They form solutions to the following coupled differential equations:*

$$\frac{1}{\hbar c}\left(E - V(r) + mc^2\right) f - \left(\frac{dg}{dr} + (1 + k)\frac{g}{r}\right) = 0$$

$$\frac{1}{\hbar c}\left(E - V(r) - mc^2\right) g + \left(\frac{df}{dr} + (1 - k)\frac{f}{r}\right) = 0 \qquad 13$$

The final electron state is similarly given by

$$\psi_{\gamma_1}(\bar{r}) = \frac{1}{r}\begin{bmatrix} C^{1p''}\, Y_{\ell''\, m''-1/2}(\theta,\phi)\, G(r) \\ C^{2p''}\, Y_{\ell''\, m''+1/2}(\theta,\phi)\, G(r) \\ C^{3p''}\, Y_{\ell''\, m''-1/2}(\theta,\phi)\, iF(r) \\ C^{4p''}\, Y_{\ell''\, m''+1/2}(\theta,\phi)\, iF(r) \end{bmatrix} \qquad 14$$

Notice that Equation 14 has a slightly different r dependence as compared to Equation 12. The $G(r)$ and $F(r)$ functions satisfy the following differential equations:

$$\frac{dF}{dr} - k\frac{F}{r} = \left[\frac{mc}{\hbar}\left(1 - \frac{E}{mc^2}\right) + \frac{V}{\hbar c}\right] G,$$

$$\frac{dG}{dr} + k\frac{G}{r} = \left[\frac{mc}{\hbar}\left(1 + \frac{E}{mc^2}\right) - \frac{V}{\hbar c}\right] F \qquad 15$$

which can be obtained from Equation 13 by substituting $G(r)/r$ for $g(r)$ and $F(r)/r$ for $f(r)$.*

*Bethe, Ibid., p. 65

Similarly, for the spin down initial electron the wave function is given by*

$$p = 1$$
$$j = 1/2,$$
$$\ell = 0,$$
$$k = 0,$$
$$m = - 1/2. \qquad 16$$

This then yields

$$\psi_{\beta_1} = \begin{bmatrix} 0 \\ C^{2+}\ Y_{00}\ g(r) \\ C^{3+}\ Y_{1\ -1}\ if(r) \\ C^{4+}\ Y_{1\ 0}\ if(r) \end{bmatrix} \qquad 17$$

<u>YETE's</u>

$$\text{YETE (1)} = \int dr'\ r'^2\ g(r')\ j_\ell(kr') \int dr\ r\ j_n(k_o r)\ G^*(r)\ RR(p'ri:rr'),$$

$$\text{YETE (2)} = \int dr'\ r'^2\ f(r')\ j_\ell(kr') \int dr\ r\ j_n(k_o r)\ G^*(r)\ RS(p'ri:rr'),$$

$$\text{YETE (3)} = \int dr'\ r'^2\ g(r')\ j_\ell(kr') \int dr\ r\ j_n(k_o r)\ F^*(r)\ RS(p'ir:r'r),$$

$$\text{YETE (4)} = \int dr'\ r'^2\ f(r')\ j_\ell(kr') \int dr\ r\ j_n(k_o r)\ F^*(r)\ SS(p'ri:rr'), \qquad 18$$

*Bethe, Ibid., p. 64.

YATE's

$$\text{YATE (1)} = \int dr'\, r'\, f(r')\, j_{\ell'}(k_o r') \int dr\, r\, j_{\ell}(kr)\, G^*(r)\, RS(p'ri:rr')$$

$$\text{YATE (2)} = \int dr'\, r'\, j_{\ell'}(k_o r')\, f(r') \int dr\, r\, j_{\ell}(kr)\, F^*(r)\, SS(p'ri:rr'),$$

$$\text{YATE (3)} = \int dr'\, r'^2\, j_{\ell'+p'}(k_o r')\, g(r') \int dr\, r\, j_{\ell}(kr)\, G^*(r)\, RR(p'ri:rr'),$$

$$\text{YATE (4)} = \int dr'\, r'^2\, j_{\ell'+p'}(k_o r')\, g(r') \int dr\, r\, j_{\ell}(kr)\, F^*(r)\, RS(p'ir:r'r),$$

$$\text{YATE (5)} = \int dr'\, r'^2\, j'_{\ell'}(k_o r')\, f(r') \int dr\, r\, j_{\ell}(kr)\, G^*(r)\, RS(p'ri:rr'),$$

$$\text{YATE (6)} = \int dr'\, r'^2\, j'_{\ell'}(k_o r)\, f(r') \int dr\, r\, j_{\ell}(kr)\, F^*(r)\, SS(p'ri:rr'). \quad 19$$

In some of the above equations one encounters j' which should be interpreted as the first derivative of the spherical Bessel function with respect to its argument.

Define the following constants:

$$A = C^{4+}\, i^{\ell+1}\, (3\ell'(\ell'+1)(2\ell'+1))^{\frac{1}{2}}\, k_o^{-1} ,$$

$$AB = C^{1+}\, i^{\ell'+p'}\, (4(\ell'+p')+2)^{\frac{1}{2}} ,$$

$$AC = C^{3+}\, i^{\ell'+1} (6(2\ell'+1))^{\frac{1}{2}} ,$$

$$B = -\, C^{2+}\, i^{\ell'+p'}\, (4(\ell'+p')+2)^{\frac{1}{2}} ,$$

$$BA = -\, C^{4+}\, i^{\ell'+1} (6(2\ell'+1))^{\frac{1}{2}} ,$$

$$BB = C^{3+}\, i^{\ell'-1} (3\ell'(\ell'+1)(2\ell'+1))^{\frac{1}{2}}\, k_o^{-1} . \quad 20$$

WETE'S

$$\text{WETEU}(1) = C^{1p''}\ C^{1+}\ I\ (\ell'',m''-1/2:\ell,m''-1/2:n,0)\ \delta_{\ell',\ell-p'}$$

$$((m''-3/2\ C(43:p') + CC(\ell,m''-3/2)\ C(44:p')),$$

$$\text{WETEU}(2) = -\ C^{1p''}\ 3^{1/2}\ \{I(\ell'',m''-1/2:\ell+p'+1,m''-1/2:n,0)$$

$$((2\ell+1)(2\ell+3))^{-1/2}\ \delta_{\ell',\ell+1}\ [(m''-3/2)\ C^{3+}\ C(41:p')$$

$$((\ell+m''-1/2)(\ell-m''+5/2))^{1/2} + C^{3+}\ CC(\ell,m''-3/2)\ C(42:p')$$

$$((\ell+m''+1/2)(\ell-m''+3/2))^{1/2} + C^{4+}\ 2^{-1/2}\ CC(\ell,-m''+3/2)\ C(41:p')$$

$$((\ell+m''-3/2)(\ell+m''-1/2)/2)^{1/2} - C^{4+}\ 2^{-1/2}\ (m''-3/2)\ C(42:p')$$

$$((\ell+m''-3/2)(\ell+m''+1/2)/2)^{1/2}]$$

$$+\ I(\ell'',m''-1/2:\ell'+p'-1,m''-1/2:n,0)((2\ell+1)(2\ell-1))^{-1/2}\delta_{\ell',\ell-1}$$

$$[C^{3+}(m''-3/2)\ C(41:p')((\ell+m''-3/2)(\ell-m''+3/2))^{1/2}$$

$$+\ C^{3+}\ CC(\ell,-m)\ C(42:p')((\ell+m''-1/2)(\ell-m''-5/2))^{1/2}$$

$$-\ C^{4+}\ CC(\ell,-m''+3/2)\ C(41:p')((\ell-m''+5/2)(\ell-m''+3/2)/2)^{1/2}$$

$$+\ C^{4+}(m''-3/2)\ C(42:p')((\ell-m''+1/2)(\ell-m''+3/2)/2)^{1/2}\ ]\},$$

$$\text{WETEU}(3) = -C^{3p''}\ C^{1+}\ I\ (\ell''+p'',m''-1/2:\ell-p',m''-1/2:n,0)\ \delta_{\ell',\ell-p'}$$

$$((m''-3/2\ C(23:p') + CC(\ell,m''-3/20\ C(24:p')),$$

$$WETEU(4) = - C^{3p''} \; 3^{1/2} \{ I(\ell''+p'',m''-1/2:\ell+1,m''-1/2:n,0)\delta_{\ell',\ell+1}$$

$$((2\ell+1)(2\ell+3))^{-1/2} [C^{3+}(m''-3/2)C(21:p')$$

$$((\ell+m''-1/2)(\ell-m''+5/2))^{1/2} + C^{3+} CC(\ell,m''-3/2) \; C(22:p')$$

$$((\ell+m''+1/2)(\ell-m''+3/2))^{1/2} + C^{4+} \; 2^{-1/2}CC(\ell-m''+3/2) \; C(21:p')$$

$$((\ell+m''-3/2)(\ell+m''-1/2)/2)^{1/2} - C^{4+} \; 2^{-1/2} \; (m''-3/2) \; C(22:p')$$

$$((\ell+m''-3/2)(\ell+m''+1/2)/2)^{1/2}]$$

$$+ I(\ell''+p'',m''-1/2:\ell-1,m''-1/2:n,0) \; \delta_{\ell',\ell-1}((2\ell+1)(2\ell-1))^{-1/2}$$

$$[C^{3+}(m''-3/2) \; C(21:p')((\ell+m''-3/2)(\ell-m''+3/2))^{1/2}$$

$$+ C^{3+} \; CC(\ell,m''-3/2) \; C(22:p')((\ell+m''-1/2)(\ell-m''+1/2))^{1/2}$$

$$- C^{4+} \; 2^{-1/2} \; CC(\ell,-m''+3/2) \; C(21:p') \; ((\ell - m''+5/2)(\ell-m''+3/2)/2)^{1/2}$$

$$+ C^{4+} \; 2^{-1/2}(m''-3/2) \; C(22:p')((\ell-m''+1/2)(\ell-m''+3/2)/2)^{1/2}] \},$$

$$WETEU(5) = C^{3p''} \; C^{1+} \; I \; (\ell'',m''+1/2:\ell,m''+1/2:n,0) \; \delta_{\ell,\ell-p'}$$

$$((m''+1/2) \; C(33:p') + CC(\ell,m''+1/2) \; C(34:p')),$$

$$WETEU(6) = - C^{3p''}\, 3^{1/2}\, \{I\,(\ell'',m''+1/2:\ell+p'+1,m''+1/2:n,0)\ \delta_{\ell',\ell+1}$$

$$((2\ell+1)(2\ell+3))^{-1/2}\,[C^{3+}(m''+1/2)\ C(31:p')(\ell+m''+3/2)(\ell-m''+1/2))^{1/2}$$

$$+ C^{3+}\ CC(\ell,m''+1/2)\ C(32:p')((\ell+m''+5/2)(\ell-m''-1/2))^{1/2}$$

$$+C^{4+}2^{-1/2}\ CC(\ell,-m''-1/2)\ C(31:p')((\ell+m''+1/2)(\ell+m''3/2)/2)^{1/2}$$

$$- C^{4+}\ 2^{-1/2}\ (m''+1/2)\ C(32:p')((\ell+m''+3/2)(\ell+m''+5/2)/2)^{1/2}]$$

$$+ I\ (\ell'',m''+1/2:\ell+p'-1,m''+1/2:n,0)\ \delta_{\ell',\ell-1}((2\ell+1)(2\ell-1)^{-1/2}$$

$$[\ C^{3+}(m''+1/2)\ C(31:p')\ ((\ell+m''+1/2)(\ell-m''-1/2))^{1/2}$$

$$+ C^{3+}\ CC(\ell,m''+1/2)\ C(32:p')\ ((\ell+m''+3/2)(\ell-m''-3/2))^{1/2}$$

$$- C^{4+}\ 2^{-1/2}\ CC(\ell,-m''-1/2)\ C(31:p')\ ((\ell-m''+1/2)(\ell-m''-1/2)/2)^{1/2}$$

$$+ C^{4+}\ 2^{-1/2}\ (m''+1/2)\ C(32:p')\ ((\ell-m''-1/2)(\ell-m''-3/2)/2)^{1/2}]\},$$

$$WETEU(7) - C^{4p''}\ C^{1+}\ I\ (\ell''+p'',m''+1/2:\ell-p',m''+1/2:n,0)\delta_{\ell',\ell-p'}$$

$$((m''+1/2)\ C(13:p') + CC(\ell,m''+1/2)\ C(14:p')),$$

$$WETEU(8) = C^{4p''}\, 3^{1/2} \{ I\,(\ell''+p'', m''+1/2 : \ell+1, m''+1/2 : n, 0)\, \delta_{\ell', \ell+1}$$

$$((2\ell+1)(2\ell+3))^{-1/2} [C^{3+}\, (m''+1/2)\, C(11:p')\, ((\ell+m''+3/2)(\ell-m''+1/2))^{1/2}$$

$$+ C^{3+}\, CC(\ell, m''+1/2)\, C(12:p')((\ell+m''+5/2)(\ell-m''-1/2))^{1/2}$$

$$+ C^{4+}\, 2^{-1/2}\, CC(\ell, -m''-1/2)\, C(11:p')((\ell+m''+1/2)(\ell+m''+3/2)/2)^{1/2}$$

$$- C^{4+}\, 2^{-1/2}\, (m''+1/2)\, C(12:p')\, ((\ell+m''+3/2)(\ell+m''+5/2)/2))^{1/2}]$$

$$+ I\,(\ell''+p'', m''+1/2 : \ell-1, m''+1/2 : n, 0)\, \delta_{\ell', \ell-1}((2\ell+1)(2\ell-1))^{-1/2}$$

$$[C^{3+}\, (m''+1/2)\, C(11:p')((\ell+m''+1/2)(\ell-m''-1/2))^{1/2}$$

$$+ C^{3+}\, CC(\ell, m''+1/2)\, C(12:p')\, ((\ell+m''+3/2)(\ell-m''-3/2))^{1/2}$$

$$- C^{4+}\, 2^{-1/2}\, CC(\ell, -m''-1/2)\, C(11:p')((\ell-m''+1/2)(\ell-m''-1/2)/2)^{1/2}$$

$$+ C^{4+}\, 2^{-1/2}\, (m''+1/2)\, C(12:p')\, ((\ell-m''-1/2)(\ell-m''-3/2)/2)^{1/2}] \},$$

$$WETED(1) = C^{1p''}\, C^{2+}\, I(\ell'', m''-1/2 : \ell, m''-1/2 : n, 0)\, \delta_{\ell', \ell-p'}$$

$$(CC(\ell, -m''+1/2)\, C(43:p') - (m''-1/2)\, C(44:p')),$$

$$\mathrm{WETED}(2) = C^{1p''}\ 3^{1/2}\ \{I\ (\ell'',m''-1/2:\ell+p'+1,m''-1/2:n,0)\ .$$

$$((2\ell+1)2\ell-1)^{-1/2}\delta_{\ell',\ell+1}[C^{3+}\ (-m''+1/2)\ 2^{-1/2}\ C(41:p')$$

$$((\ell-m''+3/2)(\ell-m''+5/2))^{1/2} - C^{3+}\ 2^{-1/2}\ CC(\ell,m''-1/2)\ C(41:p')$$

$$((\ell-m''+1/2)(\ell-m''+3/2))^{1/2} - C^{4+}\ CC(\ell,-m''+1/2)\ C41:p')$$

$$((\ell+m''-1/2)(\ell-m''+5/2))^{1/2} + C^{4+}\ (m''-1/2)\ C(42:p')$$

$$((\ell+m''+1/2)(\ell-m''+3/2))^{1/2}]$$

$$+\ I\ (\ell'',m''-1/2:\ell+p'-1,m''-1/2:n,0)((2\ell+1)(2\ell-1))^{-1/2}\ \delta_{\ell',\ell-1}$$

$$[\ C^{3+}(m''-1/2)\ C(41:p')((\ell+m''-1/2)(\ell+m''-3/2))^{1/2}$$

$$+\ C^{3+}\ CC(\ell,m''-1/2)\ C(42:p')((\ell+m''+1/2)(\ell+m''-1/2))^{1/2}$$

$$-\ C^{4+}\ 2^{1/2}\ CC\ (\ell,-m''+1/2)\ C(41:p')((\ell+m''-3/2)(\ell-m''+3/2))^{1/2}$$

$$+\ C^{4+}\ 2^{1/2}\ (m''-1/2)\ C(42:p')((\ell+m''-1/2)(\ell-m''+1/2))^{1/2}]\}\ ,$$

$$\mathrm{WETED}(3) = -\ C^{3p''}\ C^{2+}\ I(\ell''+p'',m''-1/2:\ell-p',m''-1/2:n,0)\ \delta_{\ell',\ell-p'}$$

$$(C(\ell,-m''+1/2)\ C(23:p') - (m''-1/2)\ C(24:p')),$$

$$\mathrm{WETED}(4) = C^{3p''}\ 3^{1/2}\ \{I\ (\ell''+p'',m''-1/2:\ell+1,m''-1/2:n,0)$$

$$((2\ell+1)(2\ell+3))^{-1/2}\ \delta_{\ell',\ell+1}\ [C^{3+}\ 2^{-1/2}(m''-1/2)\ C(21:p')$$

$$((\ell-m''+3/2)(\ell-m''+5/2))^{1/2} + C^{3+}\ 2^{-1/2}\ CC(\ell,m''-1/2)\ C(22:p')$$

$$((\ell-m''+1/2)(\ell-m''+3/2))^{1/2} + C^{4+}\ CC(\ell,-m''+1/2)\ C(21:p')$$

$$((\ell+m''-1/2)(\ell-m''+5/2))^{1/2} - C^{4+}\ (m''-1/2)\ C(22:p')$$

$$((\ell+m''+1/2)(\ell-m''+3/2))^{1/2}]$$

$$+ I\ (\ell''+p,m''-1/2:\ell-1,m''-1/2:n,0)((2\ell+1)(2\ell-1)^{-1/2}\ \delta_{\ell',\ell-1}$$

$$[-C^{3+}\ 2^{-1/2}\ (m''-1/2)\ C(21:p')\ ((\ell+m''-1/2)(\ell+m''-3/2))^{1/2}$$

$$- C^{3+}\ 2^{-1/2}\ CC(\ell,m''-1/2)\ C(22:p')((\ell+m''+1/2)(\ell+m''-1/2))^{1/2}$$

$$+ C^{4+}\ CC(\ell,-m''+1/2)\ C(21:p')((\ell+m''-3/2)(\ell-m''+3/2))^{1/2}$$

$$- C^{4+}(m''-1/2)\ C(22:p')((\ell+m''-1/2)(\ell-m''+1/2))^{1/2}\ ]\},$$

$$\mathrm{WETED}(5) = C^{2p''}\ C^{2+}\ I\ (\ell'',m''+1/2:\ell,m''+1/2:n,0)\ \delta_{\ell,\ell-p'}$$

$$(CC(\ell,-m''-3/2)\ C(33:p') - (m''+3/2)\ C(34:p')),$$

$$WETED(6) = -C^{2p''}\, 3^{1/2}\ \{\ I(\ell'',m''+1/2:\ell+p'+1,m''+1/2:n,o)$$

$$((2\ell+1)(2\ell+3))^{-1/2}\delta_{\ell',\ell+1}[C^{3+}2^{-1/2}(m''+3/2)\ C(31:p')$$

$$((\ell-m''-1/2)(\ell-m''+1/2))^{1/2}+C^{3+}2^{-1/2}\ CC(\ell,m''+3/2)\ C(32:p')$$

$$((\ell-m''-3/2)(\ell-m''-1/2))^{1/2}+C^{4+}\ CC(\ell,-m''-3/2)\ C(31:p')$$

$$((\ell+m''+3/2)(\ell-m''+1/2))^{1/2}-C^{4+}(m''+3/2)\ C(32:p')$$

$$((\ell+m''+5/2)(\ell-m''-1/2))^{1/2}]$$

$$+I(\ell'',m''+1/2:\ell+p'-1,m''+1/2:n,o)\ \ ((2\ell+1)\ (2\ell-1))^{-1/2}\delta_{\ell',\ell-1}$$

$$[-C^{3+}2^{-1/2}\ (m''+3/2)\ C(31:p')\ \ ((\ell+m''+3/2)\ (\ell+m''+1/2))^{1/2}$$

$$-C^{3+}2^{-1/2}\ CC(\ell,m''+3/2)\ C(32:p')\ \ ((\ell+m''+5/2)\ (\ell+m''+3/2))^{1/2}$$

$$+C^{4+}CC(\ell,-m''-3/2)\ C(31:p')\ ((\ell+m''+1/2)\ (\ell-m''-1/2))^{1/2}$$

$$-C^{4+}(m''+3/2)\ C(32:p')\ \ ((\ell+m''+3/2)\ (\ell-m''-3/2))^{1/2}]\}$$

$$WETED(7) = -C^{4p''}\ C^{2+}\ I(\ell''+p'',m''+1/2:\ \ell-p',m''+1/2:n,o)\,\delta_{\ell',\ell-p'}$$

$$(CC(\ell,-m''-3/2)\ C(13:p')\ -\ (m''+3/2)\ C(14:p')),$$

$$
\begin{aligned}
\mathrm{WETED}(8) = {} & C^{4p''}\, 3^{1/2} \{\, I(\ell''+p'',m''+1/2:\ell+1,m''+1/2:n,o) \\
& ((2\ell+1)\ (2\ell+3))^{-1/2} \delta_{\ell',\ell+1} [C^{3+} 2^{-1/2} (m''+3/2)\ C(11:p') \\
& ((\ell-m''-1/2)\ (\ell-m''+1/2))^{1/2} + C^{3+} 2^{-1/2}\ CC(\ell,m''+3/2)\ C(12:p') \\
& ((\ell-m''-3/2)\ (\ell-m''-1/2))^{1/2} + C^{4+}\ CC(\ell,-m''-3/2)\ C(11:p') \\
& ((\ell+m''+3/2)\ (\ell-m''+1/2))^{1/2} - C^{4+} (m''+3/2)\ C(12:p') \\
& ((\ell+m''+5/2)\ (\ell-m''-1/2))^{1/2}] \\
& + I(\ell''+p'',m''+1/2:\ell-1,m''+1/2:n,o)\ ((2\ell+1)\ (2\ell-1))^{-1/2} \delta_{\ell',\ell-1} \\
& [-C^{3+} 2^{-1/2}\ (m''+3/2)\ C(11:p')\ \ ((\ell+m''+3/2)\ (\ell+m''+1/2))^{1/2} \\
& -C^{3+} 2^{-1/2}\ CC(\ell,m''+3/2)\ C(12:p')\ \ ((\ell+m''+5/2)\ (\ell+m''+3/2))^{1/2} \\
& +C^{4+}\ CC(\ell,-m''-3/2)\ \ C(11:p')\ \ ((\ell+m''+1/2)\ (\ell-m''-1/2))^{1/2} \\
& -C^{4+}\ (m''+3/2)\ C(12:p')\ \ ((\ell+m''+3/2)\ (\ell-m''-3/2))^{1/2}\ ]\}. \qquad 21
\end{aligned}
$$

WATE'S

WATEU(1) = A[C^{1p''} { C(31:p') (m''-3/2)

I(ℓ'',m''-1/2:ℓ,m''-3/2:ℓ'+p',1)

-C(41:p') ((ℓ+m''-3/2) (ℓ-m''+5/2))^{1/2}

I(ℓ'',m''-1/2:ℓ,m''-5/2:ℓ'+p',2)}

-C^{2p''} { C(31:p') ((ℓ-m''+3/2) (ℓ+m''-1/2))^{1/2}

I(ℓ'',m''+1/2:ℓ,m''-1/2:ℓ'+p',1)

+C(41:p') (m''-3/2) I(p'',m''+1/2:ℓ,m''-3/2:ℓ'+p',2)}],

WATEU(2) = -A[C^{3p''} { C(11:p') (m''-3/2) I(ℓ''+p'',m''-1/2:ℓ,m''-3/2:ℓ',1)

-C(21:p') ((ℓ+m''-3/2) (ℓ-m''+5/2))^{1/2}

I(ℓ''+p'',m''-1/2:ℓ,m''-5/2:ℓ',2)}

-C^{4p''} { C(11:p') ((ℓ-m''+3/2) (ℓ+m''-1/2))^{1/2}

I(ℓ''+p'',m''+1/2:ℓ,m''-1/2:ℓ',1)

+ C(21:p') (m''-3/2) I(ℓ''+p'',m''+1/2:ℓ,m''-3/2:ℓ',2) }];

$$WATEU(3) = AB[C^{1p''} \{C(34:p')\ (m''+1/2)$$

$$I(\ell'',m''-1/2:\ell,m''+1/2:\ell'+p',-1)$$

$$- C(44:p')\ ((\ell+m''+1/2)\ (\ell-m''+1/2))^{1/2}$$

$$I(\ell'',m''-1/2:\ell,m''-1/2:\ell'+p',0)\ \}$$

$$- C^{2p''}\ \{\ C(34:p')\ ((\ell-m''-1/2)\ (\ell+m''+3/2))^{1/2}$$

$$I(\ell'',m''+1/2:\ell,m''+3/2:\ell'+p',-1)$$

$$+ C(44:p')\ (m''+1/2)\ I(\ell'',m''+1/2:\ell,m''+1/2:\ell'+p',0)\ \}],$$

$$WATEU(4) = AB[C^{3p''}\ \{\ C(24:p')\ ((\ell+m''+1/2)\ (\ell-m''+1/2))^{1/2}$$

$$I(\ell''+p'',m''-1/2:\ell,m''-1/2:\ell',0)$$

$$- C(14:p')\ (m''+1/2)\ I(\ell''+p'',m'' - 1/2:\ell,m''+1/2:\ell',-1)\}$$

$$+C^{4p''}\ \{\ C(24:p')\ (m''+1/2)\ I(\ell''+p'',m''+1/2:\ell,m''+1/2:\ell',0)$$

$$+ C(14:p')\ ((\ell-m''-1/2)\ (\ell+m''+3/2))^{1/2}$$

$$I(\ell''+p'',m''+1/2:\ell,m''+3/2:\ell',-1)\}],$$

(20)

$$WATEU(5) = AC[C^{1p''} \{ C(42:p') ((\ell+m''+1/2)\ (\ell-m''+1/2))^{1/2}$$

$$I(\ell'',m''-1/2:\ell,m''-1/2:\ell'+p',0)$$

$$- C(32:p')\ (m''+1/2)\ I(\ell'',m''-1/2:\ell,m''+1/2:\ell'+p',-1)\}$$

$$+C^{2p''} \{ C(32:p') ((\ell-m''-1/2)\ (\ell+m''+3/2)$$

$$I(\ell'',m''+1/2:\ell,m''+3/2:\ell'+p',-1)$$

$$+C(42:p')\ (m''+1/2)\ I(\ell'',m''+1/2:\ell,m''+1/2:\ell'+p',0)\ \}],$$

$$WATEU(6) = AC[C^{3p''} \{ C(12:p')\ (m''+1/2)\ I(\ell''+p'',m''-1/2:\ell,m''+1/2:\ell',-1)$$

$$-C(22:p') ((\ell+m'+1/2)\ (\ell-m''+1/2))^{1/2}$$

$$I(\ell''+p'',m''-1/2:\ell,m''-1/2:\ell',0)\}$$

$$- C^{4p''} \{ C(12:p') ((\ell-m''-1/2)\ (\ell+m''+3/2))^{1/2}$$

$$I(\ell''+p'',m''+1/2:\ell,m''+3/2:\ell',-1)$$

$$+C(22:p')\ (m''+1.2)\ I(\ell''+p'',m''+1/2:\ell,m''+1/2:\ell',0)\}],$$

$$WATED(1) = BB[C^{1p''} (-(m''+3/2)\ C(32:p')\ I(\ell'',m''-1/2:\ell,m''+3/2:\ell'+p',-2)$$

$$+ ((\ell+m''+3/2)\ (\ell-m''-1/2))^{1/2}C(42:p')$$

$$I(\ell'',m''-1/2:\ell,m''+1/2:\ell,m''+1/2:\ell'+p',-1))$$

$$+C^{2p''} (((\ell-m''-3/2)\ (\ell+m''+5/2))^{1/2}C(32:p')$$

$$I(\ell',m''+1/2:\ell,m''+1/2:\ell,m''+5/2:\ell'+p',-2)$$

$$+ (m''+3/2)\ C(42:p')\ I(\ell'',m''+1/2:\ell,m''+3/2:\ell'+p',-1))],$$

$$WATED(2) = -BB[C^{3p''}(-(m''+3/2)\ C(12:p')\ I(\ell''+p'',m''-1/2:\ell',m''+3/2:\ell',-2)$$

$$+((\ell+m''+3/2)(\ell-m''-1/2))^{1/2}\ C(22:p')$$

$$I(\ell''+p'',m''-1/2:\ell,m''+1/2:\ell',-1))$$

$$+C^{4p''}(((\ell-m''-3/2)(\ell+m''+5/2))^{1/2}C(12:p')$$

$$I(\ell''+p'',m''+1/2:\ell,m''+5/2:\ell',-2)$$

$$+(m''+3/2)\ C(22:p')\ I(\ell''+p'',m''+1/2:\ell,m''+3/2:\ell',-1))],$$

$$WATED(3) = B[C^{1p''}((-m''+1/2)\ C(33:p')\ I(\ell'',m''-1/2:\ell,m''-1.2:\ \ell'+p',0)$$

$$+((\ell+m''-1/2)(\ell-m''+3/2))^{1/2}C(43:p')$$

$$I(\ell'',m''-1/2:\ell,m''-3/2:\ell'+p',1))$$

$$+C^{2p''}(((\ell-m''+1/2)(\ell+m''+1/2))^{1/2}C(33:p')$$

$$I(\ell'',m''+1/2:\ell,m''+1/2:\ell'+p',0)$$

$$+(m''-1/2)\ C(43:p')\ I(\ell'',m''+1/2:\ell,m''-1/2:\ell'+p',1))],$$

$$WATED(4) = -B[C^{3p''}((-m''+1/2)\ C(13:p')\ I(\ell''+p'',m''-1/2:\ell,m''-1/2:\ell',0)$$

$$+((\ell+m''-1/2)(\ell-m''+3/2))^{1/2}C(23:p')$$

$$I(\ell''+p'',m''-1/2:\ell,m''-3/2:\ell',1))$$

$$+C^{4p''}(((\ell-m''+1/2)(\ell+m''+1/2))^{1/2}C(13:p')$$

$$I(\ell''+p'',m''+1/2:\ell,m''+1/2:\ell',0)$$

$$+(m''-1/2)\ C(23:p')\ I(\ell''+p'',m''+1/2:\ell,m''-1/2:\ell',1))],$$

$$\text{WATED}(5) = BA[C^{1p''}((-m''+1/2)\ C(31{:}p')$$

$$I(\ell'',m''-1/2{:}\ell,m''-1/2{:}\ell,m''-1/2{:}\ell'+p',0)$$

$$+((\ell+m''-1/2)\ (\ell-m''+3/2))^{1/2}C(41{:}p')$$

$$I(\ell'',m''-1/2{:}\ell,m''-3/2{:}\ell'+p',1))$$

$$+C^{2p''}(((\ell-m''+1/2)\ (\ell+m''+1/2))^{1/2}C(31{:}p')$$

$$I(\ell'',m''+1/2{:}\ell,m''+1/2{:}\ell'+p',0)$$

$$+\ (m''-1/2)\ C(41{:}p')\ I(\ell'',m''+1/2{:}\ell,m''-1/2{:}\ell'+p',1))],$$

$$\text{WATED}(6) = -BA[C^{3p''}(-(m''-1/2)\ C(11{:}p')\ I(\ell''+p'',m''-1/2{:}\ell,m''-1/2{:}\ell',0),$$

$$+((\ell+m''-1/2)\ (\ell-m''+3/2))^{1/2}C(21{:}p')$$

$$I(\ell''+p'',m''-1/2{:}\ell,m''-3/2{:}\ell',1))$$

$$+C^{4p''}(((\ell-m''+1/2)\ (\ell+m''+1/2))^{1/2}C(11{:}p')$$

$$I(\ell+p'',m''+1/2{:}\ell,m''+1/2{:}\ell',0)$$

$$+(m''-1/2)\ C(21{:}p')\ I(\ell''+p'',m''+1/2{:}\ell,m''-1/2{:}\ell',1))]. \qquad 22$$